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A Second-order Self-adjoint Dynamic Equation on a Time Scale

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Abstract

In this paper, we are concerned with the second-order, self-adjoint dynamic equation $[p(t)x^{\Delta}(t)]^{\nabla} + q(t)x(t) = 0$ on a time scale. Little work has been done on this equation, which combines both the delta and nabla derivatives. Here, we establish some preliminary results, including an analogue of the Lagrange Identity. We then explore zeros of solutions and disconjugacy.

1 Preliminary Results

This paper is concerned with the study of the second-order self-adjoint equation $[p(t)x^{\Delta}]^{\nabla} + q(t)x = 0$ on a time scale. It is assumed that the reader is already familiar with the basic notions of calculus on a time scale. A complete introduction to the subject can be found in [2].

Throughout, we assume that $\mathbb{T}$ is a time scale. The notation $[a, b]$ is understood to mean the real interval $[a, b]$ intersected with $\mathbb{T}$.

In this paper, we retain the original definition of the "backward graininess function", $\nu(t)$. We define

$$\nu(t) := t - \rho(t).$$

This definition is consistent with the original literature published on ∇-derivatives. It is inconsistent, however with the current work on α-derivatives. When working with α-derivatives, the α-graininess, μ_α is

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defined to be $\mu_\alpha := \alpha(t) - t$. When $\alpha(t) = \rho(t)$, then, we would have $\mu_\rho := \rho(t) - t = -\nu(t)$. This inconsistency is unfortunate, but we feel it is more important that we remain consistent with the way $\nu(t)$ was defined in previously published work. To minimize confusion, we recommend the notation $\mu_\rho(t) = \rho(t) - t$ be used in work that is to be interpreted in the more general α-derivative setting.

We are interested in the second-order self-adjoint dynamic equation

$$Lx = 0 \text{ where } Lx = [p(t)x^\Delta]^\nabla + q(t)x. \tag{1}$$

Here we assume that p is continuous, q is ld-continuous and that

$$p(t) > 0 \text{ for all } t \in \mathbb{T}.$$

Define the set $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is continuous and such that $[p(t)x^\Delta]^\nabla : \mathbb{T}^\kappa_\kappa \to \mathbb{R}$ is ld-continuous. A function $x \in \mathbb{D}$ is said to be a solution of $Lx = 0$ on $\mathbb{T}$ provided $Lx(t) = 0$ for all $t \in \mathbb{T}^\kappa_\kappa$.

Since the equation we are interested in, equation (1), contains both Δ and ∇ derivatives, we establish here some results regarding the relationship between these two types of derivatives on time scales.

One of the following results relies on L'Hôpital's rule. A version of L'Hôpital's rule involving Δ derivatives is contained in [2]. We state its analog for ∇ derivatives here.

For any right dense $t_0 \in \mathbb{T}$ and any $\varepsilon > 0$, define

$$R_\varepsilon(t_0) := \{t \in \mathbb{T} : 0 < t - t_0 < \varepsilon\}$$

and let $\overline{\mathbb{T}} := \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}$.

Theorem 1 (L'Hôpital's Rule). *Assume f and g are ∇ differentiable on $\mathbb{T}$ with*

$$\lim_{t \to t_0^+} f(t) = \lim_{t \to t_0^+} g(t) = 0 \quad \textit{for some right-dense} \quad t_0 \in \overline{\mathbb{T}}$$

Furthermore, suppose there exists $\varepsilon > 0$ such that both $g(t)$ and $g^\nabla(t)$ are either strictly positive or strictly negative for all $t \in R_\varepsilon(t_0)$ and have the same sign on $R_\varepsilon(t_0)$. Then

$$\liminf_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)} \le \liminf_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)}.$$

The proof of this theorem is nearly identical to the proof of the Δ-derivative analog established in [2], and we will not include it here.

In order to determine when the two types of derivatives may be interchanged, we need to consider some of the points in our time scale separately, so let

$$A := \{t \in \mathbb{T} \mid t \text{ is left-dense and right-scattered}\}, \quad \mathbb{T}_A := \mathbb{T} \setminus A.$$

Additionally, let

$$B := \{t \in \mathbb{T} \mid t \text{ is right-dense and left-scattered}\}, \quad \mathbb{T}_B := \mathbb{T} \setminus B.$$

Lemma 2. *If $t \in \mathbb{T}_A$ then $\sigma(\rho(t)) = t$. If $t \in \mathbb{T}_B$ then $\rho(\sigma(t)) = t$.*

Theorem 3. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and f^Δ is rd-continuous on $\mathbb{T}^\kappa$ then f is ∇-differentiable on $\mathbb{T}_\kappa$, and*

$$f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) & t \in \mathbb{T}_A \\ \lim_{s \to t^-} f^\Delta(s) & t \in A. \end{cases}$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa$ and g^∇ is ld-continuous on $\mathbb{T}_\kappa$ then g is Δ-differentiable on $\mathbb{T}^\kappa$, and

$$g^\Delta(t) = \begin{cases} g^\nabla(\sigma(t)) & t \in \mathbb{T}_B \\ \lim_{s \to t^+} g^\nabla(s) & t \in B. \end{cases}$$

We will only prove the first statement. The proof of the second statement is similar.

Proof. First, assume $t \in \mathbb{T}_A$. Then there are two cases: Either

(i) t is left-scattered, or

(ii) t is both left-dense and right-dense.

Case (i): Suppose t is left-scattered and f is Δ-differentiable on $\mathbb{T}^\kappa$. Then $\rho(t)$ is right-scattered, and

$$f^\Delta(\rho(t)) = \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)}$$

Now, as f is Δ-differentiable on $\mathbb{T}^\kappa$, f is continuous on $\mathbb{T}$. Then, since t is left-scattered, f is ∇-differentiable at t, and we see that

$$\begin{aligned} f^\Delta(\rho(t)) &= \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)} \\ &= \frac{f(t) - f(\rho(t))}{t - \rho(t)} \\ &= f^\nabla(t). \end{aligned}$$

Case (ii): Now, suppose t is both left-dense and right-dense, and $f : \mathbb{T} \to \mathbb{R}$ is continuous on $\mathbb{T}$ and Δ-differentiable at t. Since t is right-dense and f is Δ-differentiable at t, we have that

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists. But t is left-dense as well, so this expression also defines $f^\nabla(t)$, and we see that

$$\begin{aligned} f^\nabla(t) &= \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \\ &= f^\Delta(t) \\ &= f^\Delta(\rho(t)). \end{aligned}$$

So, we have established the desired result in the case where $t \in \mathbb{T}_A$.

Now suppose $t \in A$. Then t is left-dense. Hence $f^\nabla(t)$ exists provided

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists.
As t is right-scattered, we need only consider the limit as $s \to t$ from the left. Then we apply L'Hôpital's rule [2], differentiating with respect to s to get

$$\lim_{s \to t-} \frac{f(t) - f(s)}{t - s} = \lim_{s \to t-} \frac{-f^\Delta(s)}{-1} = \lim_{s \to t-} f^\Delta(s).$$

Since we have assumed that f^Δ is rd-continuous, this limit exists. Hence f is ∇-differentiable, and $f^\nabla(t) = \lim_{s \to t-} f^\Delta(t)$, as desired. □

Corollary 4. *If $t_0 \in \mathbf{T}$, and $f : \mathbb{T} \to \mathbf{R}$ is rd-continuous on $\mathbb{T}$ then $\int_{t_0}^t f(\tau)\Delta\tau$ is ∇-differentiable on $\mathbb{T}$ and*

$$\left[\int_{t_0}^t f(\tau)\Delta\tau\right]^\nabla = \begin{cases} f(\rho(t)) & \text{if } t \in \mathbb{T}_A \\ \lim_{s \to t-} f(s) & t \in A. \end{cases}$$

If $t_0 \in \mathbf{T}$, and $g : \mathbb{T} \to \mathbf{R}$ is ld-continuous on $\mathbb{T}$ then $\int_{t_0}^t g(\tau)\nabla\tau$ is Δ-differentiable on $\mathbb{T}$ and

$$\left[\int_{t_0}^t g(\tau)\nabla\tau\right]^\Delta = \begin{cases} g(\sigma(t)) & \text{if } t \in \mathbb{T}_B \\ \lim_{s \to t+} g(s) & t \in B. \end{cases}$$

2 Lagrange Identity and Corollaries

In this section, we will establish the Lagrange Identity for the dynamic equation (1), and explore some corollaries and related results. Theorem 5, and Corollary 9 were previously established in Atici and Guseinov's work [1]. Our conditions on p and q are less restrictive than Atici and Guseinov's, and our domain of interest, $\mathbf{D}$, is defined more broadly. In spite of this, however, many of the proofs contained in [1] remain valid. As this is the case, we have omitted the proofs of some of the following theorems, and refer the reader to Atici and Guseinov's work.

Theorem 5. *If $t_0 \in \mathbb{T}$, and x_0 and x_1 are given constants, then the initial value problem*

$$Lx = 0, \quad x(t_0) = x_0, \quad x^\Delta(t_0) = x_1$$

has a unique solution, and this solution exists on all of $\mathbb{T}$. [1]

Definition 6. *If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the* Wronskian *of x and y, denoted $W(x,y)(t)$ is defined by*

$$W(x,y)(t) = \begin{vmatrix} x(t) & y(t) \\ x^\Delta(t) & y^\Delta(t) \end{vmatrix} \quad \text{for } t \in \mathbb{T}^\kappa.$$

Definition 7. *If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the* Lagrange bracket *of x and y is defined by*

$$\{x; y\}(t) = p(t)W(x,y)(t) \quad \textit{for } t \in \mathbb{T}^\kappa.$$

Theorem 8 (Lagrange Identity). *If $x, y \in \mathbb{D}$, then*

$$x(t)Ly(t) - y(t)Lx(t) = \{x; y\}^\nabla(t) \quad \textit{for } t \in \mathbb{T}^\kappa_\kappa.$$

Proof. Let $x, y \in \mathbb{D}$. We have

$$\begin{aligned}
\{x; y\}^\nabla &= [pW(x,y)]^\nabla \\
&= [xpy^\Delta - ypx^\Delta]^\nabla \\
&= x^\nabla p^\rho y^{\Delta\rho} + x[py^\Delta]^\nabla - y^\nabla p^\rho x^{\Delta\rho} - y[px^\Delta]^\nabla \\
&= x^\nabla p^\rho y^\nabla + x[py^\Delta]^\nabla - y^\nabla p^\rho x^\nabla - y[px^\Delta]^\nabla \\
&= x[py^\Delta]^\nabla - y[px^\Delta]^\nabla \\
&= x([py^\Delta]^\nabla + qy) - y([px^\Delta]^\nabla + qx) \\
&= xLy - yLx,
\end{aligned}$$

where we have made use of the fact that x^Δ and y^Δ are continuous and applied Theorem 3. □

Corollary 9 (Abel's Formula). *If x, y are solutions of (1) then*

$$W(x,y)(t) = \frac{C}{p(t)} \quad \textit{for } t \in \mathbb{T}^\kappa,$$

where C is a constant. [1]

Definition 10. *Define the* inner product *of x and y on $[a,b]$ by*

$$\langle x, y\rangle := \int_a^b x(t)y(t)\nabla t.$$

Corollary 11 (Green's Formula). *If $x, y \in \mathbb{D}$ then*

$$\langle x, Ly\rangle - \langle Lx, y\rangle = [p(t)W(x,y)]_a^b.$$

Theorem 12 (Converse of Abel's Formula). *Assume u is a solution of (1) with $u(t) \neq 0$ for $t \in \mathbb{T}$. If $v \in \mathbb{D}$ satisfies*

$$W(u,v)(t) = \frac{C}{p(t)},$$

then v is also a solution of (1).

Proof. Suppose that u is a solution of (1) with $u(t) \neq 0$ for any t, and assume that $v \in \mathbb{D}$ satisfies $W(u,v)(t) = \frac{C}{p(t)}$. Then by Theorem 8, we have

$$\begin{aligned}
u(t)Lv(t) - v(t)Lu(t) &= \{u; v\}^\nabla(t) \\
u(t)Lv(t) &= [p(t)W(u,v)(t)]^\nabla \\
&= [p(t)\frac{C}{p(t)}]^\nabla \\
&= C^\nabla \\
&= 0.
\end{aligned}$$

As $u(t) \neq 0$ for any t, we can divide through by it to get

$$Lv(t) = 0 \quad \text{for } t \in \mathbb{T}_{\kappa}^{\kappa}.$$

Hence v is a solution of (1) on $\mathbb{T}$. $\square$

Theorem 13 (Reduction of Order). *Let $t_0 \in \mathbf{T}^{\kappa}$, and assume u is a solution of (1) with $u(t) \neq 0$ for any t. Then a second, linearly independent solution, v, of (1) is given by*

$$v(t) = u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s$$

for $t \in \mathbf{T}$.

Proof. By Theorem 12, we need only show that $v \in \mathbb{D}$ and that $W(u,v)(t) = \frac{C}{p(t)}$ for some constant C. Consider first

$$\begin{aligned}
W(u,v)(t) &= u(t)v^{\Delta}(t) - v(t)u^{\Delta}(t) \\
&= u(t)\left[u^{\Delta}(t)\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s + \frac{u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)}\right] \\
&\quad -u^{\Delta}(t)u(t)\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s \\
&= u(t)u^{\Delta}(t)\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s + \frac{u(t)u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)} \\
&\quad -u(t)u^{\Delta}(t)\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s \\
&= \frac{1}{p(t)}.
\end{aligned}$$

Here we have $C = 1$. It remains to show that $v \in \mathbb{D}$. We have that

$$\begin{aligned}
v^{\Delta}(t) &= u^{\Delta}(t)\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s + \frac{u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)} \\
&= u^{\Delta}(t)\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s + \frac{1}{p(t)u(t)}.
\end{aligned}$$

Since $u \in \mathbb{D}, u(t) \neq 0$ and p is continuous, we have that v^{Δ} is continuous. Next, consider

$$\begin{aligned}
[p(t)v^{\Delta}(t)]^{\nabla} &= \left[p(t)u^{\Delta}(t)\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s\right]^{\nabla} + \left[\frac{1}{u(t)}\right]^{\nabla} \\
&= [p(t)u^{\Delta}(t)]^{\nabla}\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s \\
&\quad +p^{\rho}(t)u^{\Delta\rho}(t)\left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s\right]^{\nabla} - \frac{u^{\Delta}(t)}{u(t)u^{\rho}(t)}.
\end{aligned}$$

Now, the first and last terms are ld-continuous. It is not as clear that the center term is ld-continuous. Specifically, we are concerned about whether or not the expression

$$\left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s\right]^{\nabla}$$

is ld-continuous. Note that the integrand is rd-continuous. Hence Corollary 4 applies and yields

$$\left[\int_{t_0}^{t} \frac{1}{p(\tau)u(\tau)u^{\sigma}(\tau)}\Delta\tau\right]^{\nabla} = \begin{cases} \frac{1}{p^{\rho}(t)u^{\rho}(t)u^{\sigma\rho}(t)} & \text{if } t \in \mathbb{T}_A \\ \lim_{s\to t^-} \frac{1}{p(s)u(s)u^{\sigma}(s)} & t \in A. \end{cases}$$

Simplification of this expression gives

$$\left[\int_{t_0}^{t} \frac{1}{p(\tau)u(\tau)u^{\sigma}(\tau)}\Delta\tau\right]^{\nabla} = \frac{1}{p^{\rho}(t)u^{\rho}(t)u(t)} \quad \text{for } t \in \mathbb{T}.$$

This function is ld-continuous, and so we have that $v \in \mathbb{D}$. Hence by Theorem 12, v is also a solution of (1). Finally, note that as $W(u,v)(t) = \frac{1}{p(t)} \neq 0$ for any t, u and v are linearly independent. □

3 Oscillation and Disconjugacy

In this section, we establish results concerning generalized zeros of solutions of (1), and examine disconjugacy and oscillation of solutions.

Definition 14. *We say that a solution, x, of (1) has a* generalized zero *at t if*

$$x(t) = 0$$

or, if t is left-scattered and

$$x(\rho(t))x(t) < 0.$$

Definition 15. *We say that (1) is* disconjugate *on an interval $[a,b]$ if the following hold.*

(i) If x is a nontrivial solution of (1) with $x(a) = 0$, then x has no generalized zeros in $(a,b]$.

(ii) If x is a nontrivial solution of (1) with $x(a) \neq 0$, then x has at most one generalized zero in $(a,b]$.

Definition 16. *Let $\omega = \sup \mathbb{T}$, and if $\omega < \infty$, assume $\rho(\omega) = \omega$. Let $a \in \mathbb{T}$. We say that (1) is* oscillatory *on $[a,\omega)$ if every nontrivial real-valued solution has infinitely many generalized zeros in $[a,\omega)$. We say (1) is* nonoscillatory *if it is not oscillatory.*

The following Lemma is a direct consequence of the definition of nonoscillatory.

Lemma 17. *Let $\omega = \sup \mathbb{T}$. If $\omega < \infty$, then assume $\rho(\omega) = \omega$. Let $a \in \mathbb{T}$. Then if (1) is nonoscillatory on $[a,\omega)$, there is some $t_0 \in \mathbb{T}$, $t_0 \geq a$, such that (1) has a positive solution on $[t_0,\omega)$.*

Theorem 18 (Sturm Separation Theorem). *Let u and v be linearly independent solution of (1). Then u and v have no common zeros in* $\mathbb{T}^{\kappa}$. *If u has a zero at* $t_1 \in \mathbb{T}$, *and a generalized zero at* $t_2 > t_1 \in \mathbb{T}$, *then v has a generalized zero in* $(t_1, t_2]$. *If u has generalized zeros at* $t_1 \in \mathbb{T}$ *and* $t_2 > t_1 \in \mathbb{T}$, *then v has a generalized zero in* $[t_1, t_2]$.

The proof of this theorem is directly analogous to the standard proof used in the differential equations case.

Theorem 19. *If (1) has a positive solution on an interval* $\mathcal{I} \subset \mathbb{T}$ *then (1) is disconjugate on* $\mathcal{I}$. *Conversely, if* $a, b \in \mathbb{T}_{\kappa}^{\kappa}$ *and (1) is disconjugate on* $[\rho(a), \sigma(b)] \subset \mathbb{T}$, *then (1) has a positive solution on* $[\rho(a), \sigma(b)]$.

Proof. If (1) has a positive solution, u on $\mathcal{I} \subset \mathbb{T}$, then disconjugacy follows from the Sturm Separation Theorem.

Conversely, if (1) is disconjugate on the compact interval $[\rho(a), \sigma(b)]$, then let u, v be the solutions of (1) satisfying $u(\rho(a)) = 0, u^{\Delta}(\rho(a)) = 1$ and $v(\sigma(b)) = 0, v^{\Delta}(b) = -1$. Since (1) is disconjugate on $[\rho(a), \sigma(b)]$, we have that $u(t) > 0$ on $(\rho(a), \sigma(b)]$, and $v(t) > 0$ on $[\rho(a), \sigma(b))$. Then

$$x(t) = u(t) + v(t)$$

is the desired positive solution. □

Theorem 20 (Polya Factorization). *If (1) has a positive solution, u, on an interval* $\mathcal{I} \subset \mathbb{T}$, *then for any* $x \in \mathbb{D}$, *we get the* Polya Factorization

$$Lx = \alpha_1(t)\{\alpha_2[\alpha_1 x]^{\Delta}\}^{\nabla}(t) \text{ for } t \in \mathcal{I},$$

where

$$\alpha_1 := \frac{1}{u} > 0 \quad \text{on } \mathcal{I},$$

and

$$\alpha_2 := puu^{\sigma} > 0 \quad \text{on } \mathcal{I}.$$

Proof. Assume that u is a positive solution of (1) on $\mathcal{I}$, and let $x \in \mathbb{D}$. Then by the Lagrange Identity (Theorem 8),

$$\begin{aligned} u(t)Lx(t) - x(t)Lu(t) &= \{u; x\}^{\nabla}(t) \\ Lx(t) &= \frac{1}{u(t)}\{pW(u,x)\}^{\nabla}(t) \\ &= \frac{1}{u(t)}\left\{puu^{\sigma}\left[\frac{x}{u}\right]^{\Delta}\right\}^{\nabla}(t) \\ &= \alpha_1(t)\{\alpha_2\left[\alpha_1 x\right]^{\Delta}\}^{\nabla}(t), \end{aligned}$$

for $t \in \mathcal{I}$, where α_1 and α_2 are as described in the theorem. □

Theorem 21 (Trench Factorization). *Let* $a \in \mathbb{T}$, *and let* $\omega := \sup \mathbb{T}$. *If* $\omega < \infty$, *assume* $\rho(\omega) = \omega$. *If (1) is nonoscillatory on* $[a, \omega)$, *then there is* $t_0 \in \mathbb{T}$ *such that for any* $x \in \mathbb{D}$, *we get the* Trench Factorization

$$Lx(t) = \beta_1(t)\{\beta_2[\beta_1 x]^{\Delta}\}^{\nabla}(t)$$

for $t \in [t_0,\omega)$, where $\beta_1,\beta_2 > 0$ on $[t_0,\omega)$, and

$$\int_{t_0}^{\omega} \frac{1}{\beta_2(t)}\Delta t = \infty.$$

Proof. Since (1) is nonoscillatory on $[a,\omega)$, (1) has a positive solution, u on $[t_0,\omega)$ for some $t_0 \in \mathbb{T}$. Then by Theorem 20, Lx has a Polya factorization on $[t_0,\omega)$. Thus there are functions α_1 and α_2 such that

$$Lx(t) = \alpha_1(t)\{\alpha_2[\alpha_1 x]^{\Delta}\}^{\nabla}(t) \text{ for } t \in [t_0,\omega),$$

defined as described in the preceding theorem. Now, if

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)}\Delta t = \infty,$$

then take $\beta_1(t) = \alpha_1(t)$, and $\beta_2(t) = \alpha_2(t)$, and we are done. Therefore, assume that

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)}\Delta t < \infty.$$

In this case, let

$$\beta_1(t) = \frac{\alpha_1(t)}{\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s} \text{ and } \beta_2(t) = \alpha_2(t)\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)}\Delta s$$

for $t \in [t_0,\omega)$. Note that as $\alpha_1,\alpha_2 > 0$, we have $\beta_1,\beta_2 > 0$ as well. Also,

$$\begin{aligned}
\int_{t_0}^{\omega} \frac{1}{\beta_2(t)}\Delta t &= \lim_{b\to\omega, b\in\mathbf{T}} \int_{t_0}^{b} \frac{1}{\alpha_2(t)\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)}\Delta s}\Delta t \\
&= \lim_{b\to\omega, b\in\mathbf{T}} \int_{t_0}^{b} \left[\frac{1}{\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s}\right]^{\Delta} \Delta t \\
&= \lim_{b\to\omega, b\in\mathbf{T}} \left[\frac{1}{\int_b^{\omega} \frac{1}{\alpha_2(s)}\Delta s}\right] \\
&= \infty.
\end{aligned}$$

Now let $x \in \mathbb{D}$. Then

$$[\beta_1 x]^{\Delta}(t) = \left[\frac{\alpha_1(t)x(t)}{\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s}\right]^{\Delta} = \frac{\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s[\alpha_1(t)x(t)]^{\Delta} + \alpha_1(t)x(t)\frac{1}{\alpha_2(t)}}{\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)}\Delta s}$$

for $t \in [t_0,\omega)$. So we get

$$\beta_2(t)[\beta_1(t)x]^{\Delta} = \alpha_2(t)[\alpha_1(t)x(t)]^{\Delta}\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s + \alpha_1(t)x(t)$$

for $t \in [t_0,\omega)$. Taking the ∇ -derivative of both sides gives

$$\begin{aligned}
\left\{\beta_2(t)[\beta_1(t)x(t)]^{\Delta}\right\}^{\nabla} &= \left\{\alpha_2(t)[\alpha_1(t)x(t)]^{\Delta}\right\}^{\nabla}\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s \\
&\quad + \left\{\alpha_2(t)[\alpha_1(t)x(t)]^{\Delta}\right\}^{\rho}\left[\int_t^{\omega} \frac{1}{\alpha_2(s)}\Delta s\right]^{\nabla} \\
&\quad + [\alpha_1(t)x(t)]^{\nabla}
\end{aligned}$$

for $t \in [t_0, \omega)$. We now claim that the last two terms in this expression cancel. To see this, put the expression back in terms of our positive solution u, and consider $t \in \mathbf{A}$ and $t \in \mathbb{T}_A$ separately. Careful application of Theorem 3 then shows that these terms do, in fact cancel, and we get

$$\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s.$$

It then follows that

$$\beta_1(t)\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \alpha_1(t)\left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla = Lx(t),$$

for $t \in [t_0, \omega)$ and the proof is complete. □

Theorem 22 (Recessive and Dominant Solutions). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$ the we assume $\rho(\omega) = \omega$. If (1) is nonoscillatory on $[a, \omega)$, then there is a solution, u, called a recessive solution at ω, such that u is positive on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$, and if v is any second, linearly independent solution, called a dominant solution at ω, the following hold.*

(i) $\lim_{t\to\omega-} \frac{u(t)}{v(t)} = 0$

(ii) $\int_{t_0}^\omega \frac{1}{p(t)u(t)u^\sigma(t)}\Delta t = \infty$

(iii) $\int_b^\omega \frac{1}{p(t)v(t)v^\sigma(t)}\Delta t < \infty$ *for $b < \omega$, sufficiently close, and*

(iv) $\frac{p(t)v^\Delta(t)}{v(t)} > \frac{p(t)u^\Delta(t)}{u(t)}$ *for $t < \omega$, sufficiently close.*

The recessive solution, u, is unique, up to multiplication by a nonzero constant.

The proof of this theorem is directly analogous to the standard proof used in the differential equations case.

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